Problem of the Week Archive
Summer 2025

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Week 5

Problem

The following comes from Paul Hoffman's book, "The Man Who Loved Only Numbers" (Paperback edition, 1999, p.95).

> There is a much quoted story about David Hilbert, who one day noticed that a certain student had stopped attending class. When told that the student had decided to drop mathematics to become a poet, Hilbert replied, "Good– he did not have enough imagination to become a mathematician."
>
> -Robert Osserman

Penn State Math 310, a sophomore-level course in combinatorics, is our source for this week's problem. We love combinatorial proofs because they require imagination. This week's problem is no exception.

Use a combinatorial argument (a proof that demonstrates that both sides of the equation count the same set of objects) to prove that for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}. \tag{1}$$

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Solution

Suppose that X is a finite set with n elements. Denote the power set of X, i.e. the collection of all subsets of X, by $\mathscr{P}(X)$. It is argued that both sides of (1) count the cardinality of $\mathscr{C}$ defined by

$$\mathscr{C} := \{(A,B)|A,B \in \mathscr{P}(X), |A| = 1, A \cap B = \emptyset\}, \tag{2}$$

where $|\cdot|$ is the cardinality of $\cdot$. Two procedures for constructing $\mathscr{C}$ are considered.

Procedure 1: First, select the singleton A by selecting a single element of X. There are $\binom{n}{1} = n$ ways to do this. Next, form the subset B, by selecting a subset of arbitrary size from the set X/A. Since $|A| = 1$ and $|X| = n$, $|X/A| = n - 1$. Therefore there are 2^{n-1} subsets of X/A. The multiplication rule gives $|\mathscr{C}| = n2^{n-1}$. This is the right hand side of (1).

Procedure 2: The construction is broken down into cases based on the cardinality of B. The general case is to form a pair (A,B) with $|A| = 1, A \cap B = \emptyset$, and $|B| = k - 1$, $1 \le k \le n$.

The formation of this pair is carried out in two steps. The first step is to select $\mathcal{S} \subseteq X$ with $|\mathcal{S}| = k$. There are $\binom{n}{k}$ ways this can be done. Next, A and B are constructed from $\mathcal{S}$ by selecting a single element of $\mathcal{S}$ to form A, and then setting $B = \mathcal{S}/A$. Observe that $|\mathcal{S}| = k$, $|A| = 1$, gives $|B| = |\mathcal{S}/A| = k - 1$. Thus B has the desired cardinality. There are $\binom{k}{1}$ ways to form A and only one way to form B. Hence, by the product rule, the total number of ways to construct a singleton and a subset of $k - 1$ elements which is disjoint from the singleton is $k\binom{n}{k}$. The sum rule gives $|\mathscr{C}| = \sum_{k=1}^{n} k\binom{n}{k}$, which is the left hand side of (1).

It follows that

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}. \quad \blacksquare \tag{3}$$