Problem of the Week Archive
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Week 1

Problem

We thought we would ease into the series with a Penn State Math 140, i.e. Calculus I, problem. But don't be fooled, it is a bit more challenging than a typical Calc I problem, since its solution doesn't involve identifying the type of problem and applying an established algorithmic procedure.

Suppose that for some $a < b$, f and g are continuous functions on $[a, b]$, and differentiable on (a, b). Show that if $f(a) = g(a)$ and $f'(x) < g'(x)$ on (a, b), then $f(b) < g(b)$.

Solution

Let $h(x) := g(x) - f(x)$ on $[a, b]$. The difference of two functions is continuous if the functions are continuous, and differentiable if the functions are differentiable. Hence, by the assumptions of the problem, h is continuous on $[a, b]$, and diffentiable on (a, b). Consequently h satisfies the conditions of the Mean Value Theorem (MVT). Therefore there exists a $c \in (a, b)$, such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}. \tag{1}$$

The rule for differentiation of the difference of two differentiable functions, gives $h'(c) = g'(c) - f'(c) > 0$, where the inequality follows from the assumption that

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$f'(x) < g'(x)$ on (a, b). Hence

$$\frac{h(b) - h(a)}{b - a} > 0 \tag{2}$$

From the definition of h, $h(b) = g(b) - f(b)$, furthermore $h(a) = g(a) - f(a) = 0$, since it is assumed that $f(a) = g(a)$. Observe that $a < b$ implies $b - a > 0$, thus the inequality (2) yields

$$g(b) - f(b) > 0, \tag{3}$$

which shows that $f(b) < g(b)$. ■