Problem of the Week Arhive
Summer 2025

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Week 9

Problem

Our problem this week is from Penn State Math 436, an upper level linear algebra course. This course is typically the first time that students experience the full power of abstraction in mathematics, however, in a context that is not too far removed from the concrete. Learning linear algebra at this level is like learning to swim in deep water by swimming in a pool where your feet can touch the bottom.

The choice for this weeks problem is one which illustrates the power of abstraction.

Suppose that V and W are finite vector spaces over a field $\mathbb{F}$, which may either be $\mathbb{R}$ or $\mathbb{C}$. Prove that there exists an injective linear map from V to W if and only if $\dim \mathrm{V} \leq \dim \mathrm{W}$.

Solution

Four theorems from linear algebra are used in the solution of this problem. These are, in the order of appearance in the proof:

Theorem 1. *Let $T \in \mathscr{L}(\mathrm{V}, \mathrm{W})$. Then T is injective if and only if* $\mathrm{Null}(T) = \{0\}$*;*

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Theorem 2 (The Fundamental Theorem of Linear Maps)**.** *Suppose that* V *is finite dimensional and* $T \in \mathscr{L}(\mathrm{V}, \mathrm{W})$*. Then*

$$\dim \mathrm{V} = \dim \mathrm{Null}\,(T) + \dim \mathrm{Range}\,(T); \tag{1}$$

Theorem 3. *If* V *is finite dimensional and* U *is a subspace of* V*, then* $\dim \mathrm{U} \leq \dim \mathrm{V}$*;*

and

Theorem 4. *Suppose that* V *and* W *are vector spaces, and that* V *is finite-dimensional. If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ *is a basis of* V *and* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \in \mathrm{W}$*, then there exists a unique* $T \in \mathscr{L}(\mathrm{V}, \mathrm{W})$ *such that*

$$T\mathbf{v}_i = \mathbf{w}_i \quad i = 1, 2, \ldots, n. \tag{2}$$

First, it is proven that the existence of a injective map implies that $\dim \mathrm{V} \leq \dim \mathrm{W}$. Suppose that there exists an injective $T \in \mathscr{L}(\mathrm{V}, \mathrm{W})$. The first result above gives $\mathrm{Null}\,(T) = \{0\}$, which implies that $\dim \mathrm{Null}\,(T) = 0$. Since V is finite-dimensional, The Fundamental Theorem for Linear maps is applicable. Thus

$$\dim \mathrm{V} = \dim \mathrm{Null}\,(T) + \dim \mathrm{Range}\,(T) = \dim \mathrm{Range}\,(T). \tag{3}$$

By definition, $\mathrm{Range}\,(T)$ is a subspace of W, which is finite-dimensional by assumption. Therefore, by Theorem 3,

$$\dim \mathrm{Range}\,(T) \leq \dim \mathrm{W}. \tag{4}$$

The equality (3) and the inequality (4) are combined to obtain

$$\dim \mathrm{V} \leq \dim \mathrm{W}, \tag{5}$$

This completes the proof in this direction.

To prove that $\dim \mathrm{V} \leq \dim \mathrm{W}$ is sufficient for the existence of an injective map from V to W, such a map is exhibited. Set $m = \dim \mathrm{V}$ and $n = \dim \mathrm{W}$. Let

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\} \tag{6}$$

be a basis of V, and

$$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_n\} \tag{7}$$

be a basis for W. By Theorem 4,

$$T\mathbf{v}_i = \mathbf{w}_i \quad i = 1, 2, \ldots, m. \tag{8}$$

defines a unique $T \in \mathscr{L}(\mathrm{V}, \mathrm{W})$. This is possible, since $m \leq n$. Take $\mathbf{v} \in \mathrm{Null}\,(T)$. By the definition of a basis, $\mathbf{v}$ can be written uniquely as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m, \tag{9}$$

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where $a_i \in \mathbb{F}, i = 1, 2, \ldots, m$. By (8), (9), and the linearity of T,

$$T\mathbf{v} := a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 + \cdots + a_m\mathbf{w}_m. \tag{10}$$

Since $\mathbf{v} \in \mathrm{Null}\,(T)$, $T\mathbf{v} = 0$. Consequently

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 + \cdots + a_m\mathbf{w}_m = 0 \tag{11}$$

Basis vectors are linearly independent. Hence (11)implies that $a_1 = a_2 = \cdots = a_m = 0$. Thus $\mathbf{v} = 0$. Since $\mathbf{v}$ is an arbitrary element of $\mathrm{Null}\,(T)$, $\mathrm{Null}\,(T) = \{0\}$ It follows from the first theorem above that T is injective. This completes the proof of part (a). ∎