Problem of the Week Archive
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Week 3

Problem

We turn to Calculus II, Penn State Math 141, for this week's problem. With the exception of a boundary case, which will take a bit of thought, the solution to our Week 3 problem is straightforward.

Find the values of a such that the series

$$\sum_{n=1}^{\infty} (-1)^n \left[n \sin\left(\frac{a}{n}\right)\right]^n \tag{1}$$

converges absolutely, converges conditionally, or diverges.

Solution

The series in equation (1) has the form $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n)^n$, which is a tip off for an application of the root test. Recall the root test: If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L, \tag{2}$$

then the series $\sum_{n=0}^{\infty} a_n$ will converge absolutely if $L < 1$, and it will diverge if $L > 1$. The test fails if $L = 1$.

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For the series under consideration,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \left[n \sin\left(\frac{a}{n}\right)\right]^n\right|} = \lim_{n\to\infty} \left|\frac{\sin\left(\frac{a}{n}\right)}{\frac{1}{n}}\right| = |a|\,, \tag{3}$$

where the last equality follows from the limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \tag{4}$$

which is typically proven geometrically in Calculus I. It follows immediately from the root test that (1) converges absolutely for $|a| < 1$ and it diverges for $|a| > 1$. For $|a| = 1$ the test fails, thus futher analysis is needed in this boundary case.

Whenever one is faced with a complicates summand which does not immediately suggest a test or for which the chosen test has failed, it is good practice to first apply the test for divergence. The test for divergence states, that if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$ the test fails. In this problem, one must consider

$$\lim_{n\to\infty} (-1)^n \left[n \sin\left(\frac{\pm 1}{n}\right)\right]^n = \lim_{n\to\infty} (-1)^n (\pm 1)^n \left[n \sin\left(\frac{1}{n}\right)\right]^n. \tag{5}$$

Observe that

$$a_n = (-1)^n (\pm 1)^n f\left(\frac{1}{n}\right), \quad n = 1, 2 \ldots, \tag{6}$$

where

$$f(x) = \left[\frac{\sin x}{x}\right]^{\frac{1}{x}}. \tag{7}$$

Further, the limit in (5) will equal zero if and only if

$$\lim_{x\to 0} \left[\frac{\sin x}{x}\right]^{\frac{1}{x}} = 0. \tag{8}$$

The advantage conferred by replacing the evaluation of the limit in (5) with the evaluation of the limit in (8) is that all of the techniques for evaluating indeterminate forms, which are taught in Calculus II, may be used to evaluate the limit in (8).

First, note that the limit in (8) is an indeterminate form of type 1^∞. Indeterminate forms of this type cannot be handled with L'Hôpital's rule, however, such limits can be transformed into an indeterminate form of type $\frac{0}{0}$ by taking a logarithm. Indeterminate forms of this type are amenable to L'Hôpital's rule. Let

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \ln f(x) = \lim_{x\to 0} \ln \left[\frac{\sin x}{x}\right]^{\frac{1}{x}} = \lim_{x\to 0} \frac{\ln\left[\frac{\sin x}{x}\right]}{x} \tag{9}$$

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To evaluate the limit on the far right, two applications of L'Hôpital's rule, which lead to the second and fourth equalities below, are used:

$$\begin{aligned}\lim_{x\to 0}\frac{\ln\left[\frac{\sin x}{x}\right]}{x} &= \lim_{x\to 0}\frac{\ln\left[\sin x\right]-\ln x}{x}\\ &= \lim_{x\to 0}\frac{\cot x-\frac{1}{x}}{1}\\ &= \lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}\\ &= \lim_{x\to 0}\frac{-x\sin x}{\sin x+x\cos x}\\ &= \lim_{x\to 0}\frac{-\sin x}{\frac{\sin x}{x}+\cos x}\\ &=\frac{0}{2}=0.\end{aligned} \tag{10}$$

It follows from the continuity of the exponential function that $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{g(x)} = e^0 = 1$. Thus by the test for divergence the series (1) diverges if $|a| = 1$.

In summary,

$$\sum_{n=1}^{\infty}(-1)^n\left[n\sin\left(\frac{a}{n}\right)\right]^n \tag{11}$$

converges absolutely if $|a| < 1$ and diverges if $|a| \geq 1$. ■