Problem of the Week Archive
Summer 2025

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Week 7

Problem

This week's problem comes from Penn State Math 311W, a sophomore-level discrete math course known for its concrete yet challenging problems that do not depend on knowledge of calculus.

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined recursively as follows: $F_0 = 0, F_1 = 1$, and for $n > 1$

$$F_n = F_{n-1} + F_{n-2}. \tag{1}$$

Show that every two successive terms of the Fibonacci sequence are relatively prime, that is $\gcd(F_n, F_{n+1}) = 1$ for $n \geq 0$.

Solution

The proof uses induction.

The base case is $n = 0$. In this case, the definition of the Fibonacci sequence yields

$$\gcd(F_0, F_1) = \gcd(0, 1). \tag{2}$$

Let d be a common divisor of 0 and 1. Although $d|0$ for all d, $d|1$ implies that $d = 1$. Hence $d = 1$ is the only common divisor of F_0 and F_1. In particular, there is no common divisor greater than 1, consequently $\gcd(F_0, F_1) = 1$. Thus the base case holds.

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The induction hypothesis is that $\gcd\left(F_n, F_{n+1}\right) = 1$ for some $n \geq 0$. It must be shown that the induction hypothesis implies that $\gcd\left(F_{n+1}, F_{n+2}\right) = 1$. The definition of the the Fibonacci sequence is used to write

$$\gcd\left(F_{n+1}, F_{n+2}\right) = \gcd\left(F_{n+1}, F_{n+1} + F_n\right) \tag{3}$$

Two properties of the greatest common divisor, $\gcd\left(a, b\right) = \gcd\left(a, b - a\right)$ and $\gcd\left(a, b\right) = \gcd\left(b, a\right)$, are used on the right hands side of (3) to obtain

$$\begin{aligned}
\gcd\left(F_{n+1}, F_{n+1} + F_n\right) &= \gcd\left(F_{n+1}, \left[F_{n+1} + F_n\right] - F_{n+1}\right) \\
&= \gcd\left(F_{n+1}, F_n\right) = \gcd\left(F_n, F_{n+1}\right) = 1,
\end{aligned} \tag{4}$$

where the last equality follows from the induction hypothesis. Equations (3)and (4) together give

$$\gcd\left(F_{n+1}, F_{n+2}\right) = 1. \quad \blacksquare \tag{5}$$