

Featured Problem Series Fall 2025



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Week 9

Problem

The past few weeks our Featured Problems have had somewhat complex solutions. This week we turn to Penn State Math 310, a sophomore-level course in combinatorics, for a problem with a short solution that will require a bit of creative thinking.

Before the problem is stated some background on Fibonacci sequences is needed in order to avoid confusion. In combinatorics, the Fibonacci sequence F_0, F_1, F_2, \dots is typically defined as follow: set $F_0 = 0$ and $F_1 = 1$, and for $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}.$$

Outside of combinatorics the Fibonacci sequence f_0, f_1, \dots is commonly defined by $f_0 = 0, f_1 = 1$, and for $n \geq 2, f_n = f_{n-1} + f_{n-2}$. Clearly, $F_n = f_{n+1}, n = 0, 1, \dots$

The number of ways to cover a $2 \times n$ chess board, i.e a grid of unit squares with two rows and n columns, with dominoes, i.e. a rectangle made by joining one side of a unit square to that of another unit square, is given by $F_n, n = 1, 2, \dots$. For example, a 2×1 chess board can only be covered with a single vertical domino and $F_1 = 1$, while a 2×2 chess board can be covered in one of two way, either with two vertical dominoes or with two horizontal dominoes and $F_2 = 2$.



Use the correspondence between the Fibonacci sequence and the number of coverings to prove the identity

$$F_{2n} - 1 = \sum_{k=1}^n F_{2k-1}, \quad n \geq 1. \quad (1)$$

As an aside, note that for the sequence $\{f_n\}_{n=0}^{\infty}$ this identity takes the form

$$f_{2n+1} - 1 = \sum_{k=1}^n f_{2k}, \quad n \geq 1.$$

Solution

It is argued that both sides of (1) count the number of ways to cover an $2 \times 2n, n \geq 1$ chess board with dominoes such that the covering has at least one vertical domino.

First consider the left hand side of (1). There are F_{2n} ways to cover a $2 \times 2n, n \geq 1$ chess board with dominoes, and only one covering that has no vertical dominoes. Thus the number of coverings with at least one vertical dominoes is $F_{2n} - 1, n \geq 1$.

Next, the right hand side of (1) is considered. Label the columns of the chess board 1 through $2n$ left to right. The coverings of the board with at least one vertical domino are partitioned according to the column with the first vertical domino. Hence the counting problem can be broken into cases according to the position of the first vertical domino. All dominoes to the left of the first vertical domino are horizontal, and a horizontal domino covers two columns. It follows that the first vertical domino must cover an odd numbered column. With this in mind, the general case to consider is the first vertical domino is in column $2m + 1, m = 0, 1 \dots n$. The $2m$ columns to the left of the column with the first vertical domino are covered with horizontal dominoes. This can be done in 1 way. The $2n - (2m + 1) = 2(n - m) - 1 =: 2k - 1, k = 1, 2 \dots n$ columns to the right of the first vertical domino can be covered any way. The number of such covers is $F_{2k-1}, k = 1, \dots n$. It follows from the product rule that the total number of ways to cover the $2 \times (2k - 1)$ board to the right of the first vertical domino is $1 \cdot F_{2k-1}, k = 1, \dots n$. Finally, the sum rule gives the number of coverings with at least one vertical domino as

$$\sum_{k=1}^n F_{2k-1}, \quad n \geq 1. \quad (2)$$

It has been shown that the left and right hand side of (1) both count the same collection of objects. This establishes the equality. ■