Featured Problem Series
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Week 8

Problem

This week's problem comes from George Mason University Math 625, a graduate level course in applied linear algebra.

Denote the vector space of $n \times n$ matrices with elements in $\mathbb{R}$ by $\mathbb{R}^{n\times n}$. For $\mathbf{A} \in \mathbb{R}^{n\times n}$, let a_{ij} denote the element of $\mathbf{A}$ in row $1 \le i \le n$ and column $1 \le j \le n$. The transpose of $\mathbf{A}$ is denoted $\mathbf{A}^T$.

A norm $\|\cdot\|_*$ over $\mathbb{R}^n$ induces a norm over $\mathbb{R}^{n\times n}$ called the **induced norm**, which, in a slight abuse of notation, is also denoted by $\|\cdot\|_*$, and is given by

$$\|\mathbf{A}\|_* := \sup\left\{\|\mathbf{A}\mathbf{u}\|_* \middle| \mathbf{u} \in \mathbb{R}^n, \|\mathbf{u}\|_* = 1\right\} \tag{1}$$

Consider the norms over $\mathbb{R}^n$ defined by $\|\mathbf{x}\|_1 := \sum_{i=1}^n |x_i|$, $\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$, and $\|\mathbf{x}\|_\infty := \max_{1\le i\le n} |x_i|$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. For each of the following, either prove that it is an identity or give a counterexample.

(a) $\|\mathbf{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$;

(b) $\|\mathbf{A}\|_2 = \max_{1\le j\le n} \sqrt{\sum_{i=1}^n |a_{ij}|^2}$; and

(c) $\|\mathbf{A}\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|$.

Solution

Both (a) and (c) are well-known identities. On the other hand the standard identity for norm induced by the Euclidean norm is

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}} \tag{2}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{A}^T\mathbf{A}$. In fact, right hand side of (2) is the spectral norm of $\mathbf{A}$, which is commonly denoted by $\sigma_{\max}(\mathbf{A})$.

First, proofs of (a),(c), and, for completeness, (2) are given. Then a counterexample is exhibited to show that (b) is not an identity. The strategy of proof is the same for all three norms:

1. Find an upper bound on $\|\mathbf{Au}\|$ for all vectors $\mathbf{u}\in\mathbb{R}^n$ satisfying $\|\mathbf{u}\|=1$;
2. Show that the upper bound is achieved for some vector $\mathbf{e}\in\mathbb{R}^n$ satisfying $\|\mathbf{e}\|=1$;
3. Conclude that the least upper bound is equal to the upper bound found in the first step.

The starting point for the proof of (a) is to write $\mathbf{A}\in\mathbb{R}^{n\times n}$ in terms of its column vectors, i.e. $\mathbf{A}=(\mathbf{A}_1,\mathbf{A}_2,\ldots,\mathbf{A}_n), \mathbf{A}_i\in\mathbb{R}^n, i=1,2,\ldots n$. It then follows that for all $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^n$

$$\mathbf{Ax} = \sum_{i=1}^{n} x_i\mathbf{A}_i. \tag{3}$$

Given $\mathbf{u}\in\mathbb{R}^n$ satisfying $\|\mathbf{u}\|_1=1$, the identity (3) is used along with the triangle inequality at (4), and the scaler multiplication property of norms at (5) to obtain the following upper bound,

$$\|\mathbf{Au}\|_1 = \left\|\sum_{i=1}^{n} u_i\mathbf{A}_i\right\|_1 \le \sum_{i=1}^{n}\|u_i\mathbf{A}_i\|_1 \tag{4}$$

$$= \sum_{i=1}^{n}|u_i|\,\|\mathbf{A}_i\|_1 \tag{5}$$

$$\le \sum_{i=1}^{n}|u_i|\max_{1\le i\le n}\|\mathbf{A}_i\|_1 \tag{6}$$

$$= \|\mathbf{u}\|_1\max_{1\le i\le n}\|\mathbf{A}_i\|_1 = \max_{1\le i\le n}\|\mathbf{A}_i\|_1. \tag{7}$$

Since the set is finite there exists an index k such that

$$\|\mathbf{A}_k\|_1 = \max_{1\le i\le n}\|\mathbf{A}_i\|_1. \tag{8}$$

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It follows from (8) and (7) that for any $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\|_1 = 1$, that

$$\|\mathbf{A}\mathbf{u}\|_1 \leq \max_{1\leq i\leq n} \|\mathbf{A}_i\|_1 = \|\mathbf{A}_k\|_1 . \tag{9}$$

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard basis of $\mathbb{R}^n$. Note that $\|\mathbf{e}_i\| = 1, i = 1, 2, \ldots, n$. One has

$$\mathbf{A}\mathbf{e}_k = \mathbf{A}_k. \tag{10}$$

It follows that the the upper bound in (9) is achieved. Therefore

$$\|\mathbf{A}\|_1 = \max_{1\leq i\leq n} \|\mathbf{A}_i\|_1 . \tag{11}$$

Now attention is turned to (c). The starting point for this proof is to write $\mathbf{A} \in \mathbb{R}^{n\times n}$ in terms of its row vectors, that is

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^n \end{pmatrix}. \tag{12}$$

where $\left(\mathbf{A}^i\right)^T \in \mathbb{R}^n, i = 1, 2 \ldots, n$. Then for any $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} \mathbf{A}^1\mathbf{x} \\ \mathbf{A}^2\mathbf{x} \\ \vdots \\ \mathbf{A}^n\mathbf{x} \end{pmatrix}. \tag{13}$$

Thus for $\mathbf{u} = (u_1, u_2 \ldots, u_n)^T \in \mathbb{R}^n$ satisfying $\|\mathbf{u}\|_\infty = 1$, the identity (13) is used, along with the triangle inequality at (15), and the bound $|u_i| \leq 1, i = 1, 2, \ldots, n$ at (16), to obtain the following upper bound.

$$\|\mathbf{A}\mathbf{u}\|_\infty = \max_{1\leq i\leq n} \left|\mathbf{A}^i\mathbf{u}\right| = \max_{1\leq i\leq n} \left|\sum_{j=1}^n a_{ij}u_j\right| \tag{14}$$

$$\leq \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}u_j| \tag{15}$$

$$= \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|\,|u_j| \leq \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}| , \tag{16}$$

Once again, note that a finite set of real numbers has a maximum element. Hence

$$\|\mathbf{A}\mathbf{u}\|_\infty \leq \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{kj}| , \tag{17}$$

for some $1 \leq k \leq n$. It will be shown that this maximum is achieved for the vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{R}^n$ defined by

$$v_j := \begin{cases} 1 & \text{if } a_{kj} \geq 0 \\ -1 & \text{if } a_{kj} < 0. \end{cases} \tag{18}$$

The definition of absolute value yields

$$\sum_{j=1}^n |a_{kj}| = \sum_{j=1}^n a_{kj} v_j = \left|\mathbf{A}^k \mathbf{v}\right|, \tag{19}$$

where the second equality follows from the fact that all of the terms in the sum are non-negative. For $i \neq k$, the triangle inequality, equality (19), and inequality (17) give

$$\left|\mathbf{A}^i \mathbf{v}\right| = \left|\sum_{j=1}^n a_{ij} v_j\right| \leq \sum_{j=1}^n |a_{ij}| \, |v_j| = \sum_{j=1}^n |a_{ij}| \leq \left|\mathbf{A}^k \mathbf{v}\right|. \tag{20}$$

Consequently

$$\|\mathbf{A}\mathbf{v}\|_\infty = \max_{1 \leq i \leq n} \left|\mathbf{A}^i \mathbf{v}\right| = \left|\mathbf{A}^k \mathbf{v}\right| = \sum_{i=1}^n |a_{kj}| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|. \tag{21}$$

Since the upper bound in (16) is achieved for $\mathbf{v}$, it is the least upper bound over $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\|_\infty = 1$. Therefore

$$\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|.$$

Now, (2) is shown to hold. Note that for any $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\mathbf{x}^T \mathbf{x}}. \tag{22}$$

Take $\mathbf{u} \in \mathbb{R}^n$ satisfying $\|\mathbf{u}\|_2 = 1$. One has

$$\|\mathbf{A}\mathbf{u}\|_2 = \sqrt{\mathbf{u}^T \left[\mathbf{A}^T \mathbf{A}\right] \mathbf{u}}. \tag{23}$$

The matrix $\mathbf{A}^T \mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric and positive semi-definite. It follows that $\mathbb{R}^n$ has an orthonormal basis made up of eigenvectors of $\mathbf{A}^T \mathbf{A}$. These vectors are denoted by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ with their corresponding eigenvalues denoted $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. One may write

$$\mathbf{u} = \sum_{i=1}^n a_i \mathbf{u}_i, \tag{24}$$

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where

$$\sum_{i=1}^{n} a_i^2 = 1. \tag{25}$$

Consequently

$$\|\mathbf{Au}\|_2 = \sqrt{\sum_{i=1}^{n} a_i^2 \lambda_i} \le \sqrt{\lambda_n}. \tag{26}$$

However, $\|\mathbf{Au}_n\| = \sqrt{\lambda_n}$. Since the maximum is achieved,

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}}, \tag{27}$$

where $\lambda_{\max} := \lambda_n$.

Finally, a matrix is exhibited which fails to satisfy (b). Let

$$\mathbf{A} = \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ 0 & 0 \end{pmatrix}. \tag{28}$$

It is straightforward to show that

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & \sqrt{6} \\ \sqrt{6} & 3 \end{pmatrix}, \tag{29}$$

which has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$. Therefore, by (2),

$$\|\mathbf{A}\|_2 = \sqrt{5}. \tag{30}$$

On the other hand, the right hand side of (b) gives

$$\max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |a_{ij}|^2} = \max\left\{\sqrt{2}, \sqrt{3}\right\} = \sqrt{3} \ne \sqrt{5} = \|\mathbf{A}\|_2. \tag{31}$$

It follows that (b) is not an identity. ■