Featured Problem Series
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Week 7

Problem

Penn State Math 403, an upper-level real analysis course, is a rich source for challenging problems. So, we once again turn to it for our Featured Problem.

Before stating the problem, the convention for the expression of rational number as the ratio of integers is set. For $x = \frac{a}{b} \in \mathbb{Q}$, it is assumed that $a, b \in \mathbb{Z}$ are relatively prime, and that $b > 0$.

Define $f_n|\mathbb{R} \to \mathbb{R}$, $g_n|\mathbb{R} \to \mathbb{R}$, and $h_n|\mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := x\left(1 + \frac{1}{n}\right); \tag{1}$$

$$g_n(x) := \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \cup \{0\} \\ b + \frac{1}{n} & \text{if } x = \frac{a}{b} \in \mathbb{Q}\backslash\{0\}; \end{cases} \tag{2}$$

and

$$h_n(x) := f_n(x)g_n(x), \tag{3}$$

$n = 1, 2, \ldots$.

(a) Prove that $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ both converge uniformly on any bounded interval.

(b) Prove that $\{h_n\}_{n=1}^{\infty}$ does not converge uniformly on any bounded interval.

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Solution

The solution will use the following definition of uniform convergence. Consider $F_n|S \to \mathbb{R}$, $n = 1, 2, \ldots$ and $F|S \to \mathbb{R}$. The sequence of functions $\{F_n\}_{n=1}^{\infty}$ converges uniformly on S to F if and only if

$$\lim_{n\to\infty} \sup_{x\in S} |F_n(x) - F(x)| = 0. \tag{4}$$

The sequences of functions are given in the statement of the problem, but the limiting functions of those sequences are not given. Thus to use the definition one must first determine the limiting functions. To this end, suppose that $F_n \to F$ uniformly on S as $n \to \infty$. Then for any $x' \in S$

$$|F_n(x') - F(x')| \leq \sup_{x\in S} |F_n(x) - F(x)|. \tag{5}$$

If follows from the squeeze theorem that the pointwise limit of the sequence of function is the same as the uniform limit provided the latter exists. Consequently the first step in the proof is to find the pointwise limits of each of the sequences.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$,

$$f(x) = \lim_{n\to\infty} f_n(x) = x; \tag{6}$$

$$g(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \cup \{0\} \\ b & \text{if } x = \frac{a}{b} \in \mathbb{Q}\backslash\{0\}. \end{cases} \tag{7}$$

With x fixed ,$\{f_n(x)\}_{n=1}^{\infty}$, $\{g_n(x)\}_{n=1}^{\infty}$, and $\{h_n(x)\}_{n=1}^{\infty}$ are numerical sequences. For numerical sequences, the limit of a product is the product of the limits, provided each limit exists, therefore $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} f_n(x) \lim_{n\to\infty} g_n(x) = f(x)g(x)$, which gives

$$h(x) = \lim_{n\to\infty} h_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \cup \{0\} \\ a & \text{if } x = \frac{a}{b} \in \mathbb{Q}\backslash\{0\}. \end{cases} \tag{8}$$

The proof of (a) is straightforward. Let I denote a bounded interval, and set $M = \sup_{x\in I} |x| < \infty$. Then

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = \lim_{n\to\infty} \sup_{x\in I} \frac{|x|}{n} = \lim_{n\to\infty} \frac{M}{n} = 0, \tag{9}$$

and

$$\lim_{n\to\infty} \sup_{x\in I} |g_n(x) - g(x)| = \lim_{n\to\infty} \frac{1}{n} = 0. \tag{10}$$

Thus by the definition of uniform convergence both sequences of functions $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ converge uniformly on any bounded interval I.

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The proof of part (b) is more intricate than the proof of (a). It will be shown that for any bounded interval I

$$\sup_{x\in I} |h_n(x) - h(x)| = \infty, \tag{11}$$

thus the $\{h_n\}_{n=1}^{\infty}$ does not converge uniformly on any bounded interval.

To this end, observe that

$$|h_n(x) - h(x)| = \begin{cases} x\left(1+\frac{1}{n}\right)\frac{1}{n} & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \cup \{0\} \\ \frac{a}{n} + \frac{a}{b}\left(1+\frac{1}{n}\right)\frac{1}{n} & \text{if } x = \frac{a}{b} \in \mathbb{Q}\backslash\{0\}. \end{cases} \tag{12}$$

If $A \subseteq B \subset \mathbb{R}$, then $\sup A \leq \sup B$. Therefore

$$\begin{aligned} \sup_{x\in I} |h_n(x) - h(x)| &\geq \sup_{x\in I\cap\mathbb{Q}\backslash\{0\}} |h_n(x) - h(x)| \\ &= \sup_{\frac{a}{b}\in I\cap\mathbb{Q}\backslash\{0\}} \frac{\left|a + \frac{a}{b}\left(1+\frac{1}{n}\right)\right|}{n} \\ &= \sup_{\frac{a}{b}\in I\cap\mathbb{Q}\backslash\{0\}} \frac{|a|\left(1+\frac{1}{b}\left(1+\frac{1}{n}\right)\right)}{n} \\ &\geq \sup_{\frac{a}{b}\in I\cap\mathbb{Q}\backslash\{0\}} \frac{|a|}{n}, \end{aligned} \tag{13}$$

where the second inequality follow from $b > 0$.

It is now shown that for any interval I

$$\sup\left\{b \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} = \infty, \tag{14}$$

which in turn is used to prove that

$$\sup\left\{|a| \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} = \infty. \tag{15}$$

Take $\frac{p}{q} \in \text{Int}(I) \cap \mathbb{Q}\backslash\{0\}$ and $N \in \mathbb{N}$. By the density of the rationals, the set $J_N := \left(\frac{p}{q} - \frac{1}{qN}, \frac{p}{q} + \frac{1}{qN}\right) \cap \mathbb{Q}\backslash\{0\}$ is countably infinite, in particular it is not empty. For any $\frac{r}{s} \in J_N$,

$$0 < \left|\frac{r}{s} - \frac{p}{q}\right| = \frac{|rq - sp|}{sq} < \frac{1}{qN}. \tag{16}$$

From the lower bound one has $|rq - sp| > 0$. However, $rq - sp$ is an integer, thus the bound implies $|rq - sp| \geq 1$. It follows from (16), that given a $N \in \mathbb{N}$, all $\frac{r}{s} \in J_N$ satisfy

$$s > N|rq - sp| \geq N. \tag{17}$$

For N sufficiently large, $J_N \subseteq I \cap \mathbb{Q}\backslash\{0\}$, since $\frac{p}{q}$ is an interior point of I. Thus for N sufficiently large, for any $\frac{r}{s} \in J_N \subseteq I \cap \mathbb{Q}\backslash\{0\}$, $s > N$. Consequently the set $\left\{b \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\}$ is unbounded above, therefore

$$\sup\left\{b \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} = \infty. \tag{18}$$

The proof of the equality in (15) is broken into three cases.
Case I: $\inf I \geq 0$

The inequality (16) is written as

$$0 < \frac{p}{q} - \frac{1}{qN} < \frac{r}{s} < \frac{p}{q} + \frac{1}{qN},$$

The bound away from zero is obtained as follows. Since $\frac{p}{q} \in I \cap \mathbb{Q}\backslash\{0\} \subset (0, \infty)\,, p > 0$. Thus $\frac{p}{q} \geq \frac{1}{q} > \frac{1}{qN}$. From the lower bound one has for all $\frac{r}{s} \in J_N$

$$r > s\left[\frac{pN-1}{qN}\right] > N\left[\frac{pN-1}{qN}\right] = \frac{pN-1}{q}, \tag{19}$$

where the second inequality is a consequence of $s > N$ for all $\frac{r}{s} \in J_N$. As already noted, for N sufficiently large $J_N \subset I \cap \mathbb{Q}\backslash\{0\}$, hence the set $\left\{|a| \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\}$ is unbounded above. Therefore

$$\sup\left\{|a| \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} = \infty. \tag{20}$$

Case II: $\inf I < 0, \sup I > 0$

In this case,

$$\left\{|a| \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} \supset \left\{|a| \middle| x = \frac{a}{b} \in I \cap [0,\infty) \cap \mathbb{Q}\backslash\{0\}\right\} \tag{21}$$

Consequently

$$\sup\left\{|a| \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} \geq \sup\left\{|a| \middle| x = \frac{a}{b} \in I \cap [0,\infty) \cap \mathbb{Q}\backslash\{0\}\right\} = \infty, \tag{22}$$

where the equality follows from Case I.
Case III: $\sup I \leq 0$

For a set $A \subset \mathbb{R}$, let $-A = \{-x | x \in A\}$. With this convention, $a \in I \subset (-\infty, 0)$ if and only if $-a \in -I \subset [0, \infty)$. The definition of absolute value yields

$$\begin{aligned}\left\{|a| \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} &= \left\{-a \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\}\right\} \\ &= \left\{-a \middle| x = \frac{-a}{b} \in -I \cap \mathbb{Q}\backslash\{0\}\right\} \\ &= \left\{|a| \middle| x = \frac{-a}{b} \in -I \cap \mathbb{Q}\backslash\{0\}\right\} \\ &= \left\{|a'| \middle| x = \frac{a'}{b} \in -I \cap \mathbb{Q}\backslash\{0\}\right\}.\end{aligned} \tag{23}$$

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It is straightforward to show with the aid of the definition of an interval that $-I$ is an interval. It follows from Case I that

$$\sup\left\{|a|\,\middle|\,x=\frac{a}{b}\in I\cap\mathbb{Q}\backslash\{0\}\right\}=\sup\left\{|a'|\,\middle|\,x=\frac{a'}{b}\in -I\cap\mathbb{Q}\backslash\{0\}\right\}=\infty. \tag{24}$$

Thus by the bound (13)

$$\sup_{x\in I}|h_n(x)-h(x)|\geq \sup_{\frac{a}{b}\in I\cap\mathbb{Q}\backslash\{0\}\backslash\{0\}}\frac{|a|}{n}=\infty. \tag{25}$$

■

Remark: An astute reader may have noted that within the above solution it has been proven that if $f_n\to f$ and $g_n\to g$ pointwise as $n\to\infty$, then $f_ng_n\to fg$ pointwise as $n\to\infty$. On the other hand, part (b) shows that this property does not necessarily hold for uniform convergence. The following theorem gives a sufficient condition for the extension of this property to uniform convergence.

Theorem 1. *Suppose that $f_n|S\to\mathbb{R}$ and $g_n|S\to\mathbb{R}$, $n=1,2,\ldots$ converge uniformly to $f|S\to\mathbb{R}$ and $g|S\to\mathbb{R}$, respectively, on S. If $K_n:=\sup_{x\in S}|f_n(x)|<\infty$ and $M_n:=\sup_{x\in S}|g_n(x)|<\infty$ for $n=1,2,\ldots$, then f_ng_n converges uniformly to fg on S as $n\to\infty$.*

Proof. First, it is argued that $L:=\sup_{x\in S}|f(x)|<\infty$, $K:=\sup_{n\in\mathbb{N}}K_n<\infty$ and $M:=\sup_{n\in\mathbb{N}}M_n<\infty$, where the latter two bounds mean that two sequences of functions are uniformly bounded.

Consider

$$\begin{aligned}L=\sup_{x\in S}|f(x)| &=\sup_{x\in S}|f(x)-f_n(x)+f_n(x)|\\ &\leq\sup_{x\in S}|f_n(x)-f(x)|+\sup_{x\in S}|f_n(x)|,\end{aligned} \tag{26}$$

where the inequality is a consequence of the triangle inequality followed by applications of the properties of the supremum. Uniform convergence implies that for $\epsilon=1$, there exists an $N\in\mathbb{N}$

$$\sup_{x\in S}|f_n(x)-f(x)|<1 \tag{27}$$

for $n\geq N$. This along with the fact that $\sup_{x\in S}|f_N(x)|=K_N<\infty$ gives

$$L<1+K_N<\infty. \tag{28}$$

Next the uniform bounds on the sequences of functions are considered. It will suffice to show that $K<\infty$, since the proof of $M<\infty$ is identical. The starting point is similar to (26).

$$\begin{aligned} K_n = \sup_{x\in S} |f_n(x)| &= \sup_{x\in S} |f_n(x) - f(x) + f(x)| \\ &\leq \sup_{x\in S} |f_n(x) - f(x)| + \sup_{x\in S} |f(x)| \\ &< \sup_{x\in S} |f_n(x) - f(x)| + L \\ &< 1 + L, \end{aligned} \tag{29}$$

whenever $n \geq N$. Therefore

$$K = \max\{K_1, K_2, \ldots, K_{N-1}, L+1\} < \infty. \tag{30}$$

With these results in hand, the theorem can be proven. The same reasoning used in (26) gives

$$\begin{aligned} &\sup_{x\in S} |f_n(x)g_n(x) - f(x)g(x)| = \\ &\sup_{x\in S} |(f_n(x) - f(x))g_n(x) + (g_n(x) - g(x))f(x)| \\ &\leq M \sup_{x\in S} |f_n(x) - f(x)| + L \sup_{x\in S} |g_n(x) - g(x)|, \end{aligned} \tag{31}$$

It follows from the the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ to f and g, respectively, on S, and the squeeze theorem that

$$\lim_{n\to\infty} \sup_{x\in S} |f_n(x)g_n(x) - f(x)g(x)| = 0. \tag{32}$$

□

Remark: Finally, circling back to the problem, the failure of the sequence of functions $\{h_n\}_{n=1}^{\infty}$ to converge uniformly does not contradict this theorem, since for g_n as given in (2)

$$\begin{aligned} M_n = \sup_{x\in I} |g_n(x)| &\geq \sup\left\{ b + \frac{1}{n} \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\} \right\} \\ &\geq \sup\left\{ b \middle| x = \frac{a}{b} \in I \cap \mathbb{Q}\backslash\{0\} \right\} = \infty, \end{aligned} \tag{33}$$

where the first inequality is a consequence of $\mathbb{R} \supset A \supset B$ implying $\sup A \geq \sup B$, the second inequality follows from $b + \frac{1}{n} > b$, and the equality comes from equation (14).