Featured Problem Series
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Week 6

Problem

This week we turn to Penn State Math 435,an upper-level abstract algebra course, for our problem.

Consider the groups $(\mathbb{R}, +), (\mathbb{R}_{\neq 0}, \cdot)$, and $(\mathbb{R}_{>0}, \cdot)$, the reals under addition, the nonzero reals under multiplication, and the positive reals under multiplication, respectively.

Which pairs, if any, of these groups are isomorphic?

Solution

Two groups $(G, \star)$ and $(H, *)$ are **isomorphic**, denoted $(G, \star) \cong (H, *)$ or simply $G \cong H$, when no confusion about the group operations is likely to arise, if there exists a bijection $\phi | G \to H$ that satisfies

$$\phi(g_1 \star g_2) = \phi(g_1) * \phi(g_2), \tag{1}$$

for all $g_1, g_2 \in G$. The map ϕ is called an **isomorphism**.

The direct way to prove that two groups are isomorphic is to exhibit an isomorphism between the group. To prove that two groups are not isomorphic it suffices to show that one group has a property preserved by isomorphisms-such properties are called **structural**- that the other group does not have. An

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example of a structural property is group order, which is the cardinality of the group, and thus preserved by a bijection.

The indirect approach for showing two groups are isomorphic or not is to exploit the fact that $\cong$ is an equivalence relation. Therefore the collection of all groups is partitioned in to equivalence classes of isomorphic groups. Hence when considering three groups, G, H, and L, if it is shown that $G \cong H$, and $H \cong L$, it follows that $G \cong L$. On the other hand, if it is shown that $G \cong H$, and $H \ncong L$, then $G \ncong L$.

The direct approach is used to show that $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$, the structural approach is employed to show that $(\mathbb{R}, +) \ncong (\mathbb{R}_{\neq 0}, \cdot)$. It will then follow by the fact that $\cong$ is an equivalence relation that $(R_{>0}, \cdot) \ncong (\mathbb{R}_{\neq 0}, \cdot)$.

Note that $\phi|\mathbb{R} \to \mathbb{R}_{>0}$ defined by $\phi(x) := e^x$ for $x \in \mathbb{R}$ has inverse $\phi^{-1}\big|\mathbb{R}_{>0} \to \mathbb{R}$ given by $\phi^{-1}(x) = \ln x$. Thus it is a bijection. It remains to show that (1) holds for this function. For all $x, y \in \mathbb{R}$,

$$\phi(x + y) = e^{(x+y)} = e^x \cdot e^y = \phi(x) \cdot \phi(y). \tag{2}$$

Hence ϕ is an isomorphism and $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$.

As noted above the order of the group is structural. Another structural property is the order of individual elements of a group. Recall that the order of $g \in G$ is an $n \in \mathbb{N}$ such that $g^n = e$ and $g^k \neq e$ for all $k < n$, where $e \in G$ is the identity. If no such n exists, then the element has infinite order. An isomorphism $\phi|G \to H$ maps the identity in G to the identity in $e' \in H$. For $g \in G$, let $h = \phi(g) \in H$. If g has order $n < \infty$, then

$$e' = \phi(e) = \phi(g^n) = (\phi(g))^n = h^n. \tag{3}$$

On the other hand for $k < n$, since ϕ is injective

$$e' = \phi(e) \neq \phi(g^k) = (\phi(g))^k = h^k. \tag{4}$$

Consequently h also has order n. If g has infinite order, then $g^n \neq e$ for all $n \in \mathbb{N}$, and injectivity gives

$$e' = \phi(e) \neq \phi(g^n) = h^n, \tag{5}$$

for all n. Thus h also has infinite order.

The identity in $(\mathbb{R}_{\neq 0}, \cdot)$ is 1. Consider -1 in $(\mathbb{R}_{\neq 0}, \cdot)$, $-1 \cdot -1 = (-1)^2 = 1$,and $1 \neq -1 = (-1)^1$. Thus -1 has order $n = 2$ in $(\mathbb{R}_{\neq 0}, \cdot)$. However, all elements of $(\mathbb{R}, +)$, with the exception of the identity which has order 1, have infinite order. In particular, there are no elements of order 2. Consequently $(\mathbb{R}, +) \ncong (\mathbb{R}_{\neq 0}, \cdot)$.

It has been shown that $(\mathbb{R}_{>0}, \cdot) \cong (\mathbb{R}, +)$ and $(\mathbb{R}, +) \ncong (\mathbb{R}_{\neq 0}, \cdot)$. Thus by the indirect approach discussed above $(\mathbb{R}_{>0}, \cdot) \ncong (\mathbb{R}_{\neq 0}, \cdot)$.

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