Problem of the Week
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Week 5

Problem

We stay with Penn State Math 403, an upper-level undergraduate real analysis course, for this week's Featured Problem. The proof of this classic theorem in real analysis will require some creative thinking, but it does not have as many parts, and it is not as involved as the solution to last week's problem.

Prove that all norms on $\mathbb{R}^n$ are equivalent.

Solution

Consider a vector space V and $\mathcal{N}$, the collection of all norms on V. Define a relation $\sim$ on $\mathcal{N}$ as follows $\|\cdot\|' \sim \|\cdot\|$, if there exist real numbers m, M satisfying $0 < m \leq M < \infty$ such that

$$m\,\|\mathbf{v}\| \leq \|\mathbf{v}\|' \leq M\,\|\mathbf{v}\| \tag{1}$$

for all $\mathbf{v} \in \mathrm{V}$. Observe that

(i) Since $\|\mathbf{v}\| = \|\mathbf{v}\|$, $\|\mathbf{v}\| \sim \|\mathbf{v}\|$ with $m = M = 1$. Thus the relation is **reflexive**.

(ii) On the other hand, if $\|\cdot\|' \sim \|\cdot\|$, then

$$\frac{1}{M}\,\|\mathbf{v}\|' \leq \|\mathbf{v}\| \leq \frac{1}{m}\,\|\mathbf{v}\|'. \tag{2}$$

Consequently $\|\cdot\| \sim \|\cdot\|'$. Hence the relation is **symmetric**.

(iii) Finally, suppose that $\|\cdot\|' \sim \|\cdot\|$ and $\|\cdot\| \sim \|\cdot\|^\star$. Then for some $0 < m \leq M < \infty$ and $0 < l \leq L < \infty$

$$m\,\|\mathbf{v}\| \leq \|\mathbf{v}\|' \leq M\,\|\mathbf{v}\| \qquad \text{and} \qquad l\,\|\mathbf{v}\|^\star \leq \|\mathbf{v}\| \leq L\,\|\mathbf{v}\|^\star . \tag{3}$$

It follows that

$$ml\,\|\mathbf{v}\|^\star \leq \|\mathbf{v}\|' \leq ML\,\|\mathbf{v}\|^\star . \tag{4}$$

Thus $\|\cdot\|' \sim \|\cdot\|^\star$. Therefore the relation is **transitive**.

Taken together this means that $\sim$ is an **equivalence relation**. Hence two norms satisfying (1) are called **equivalent**.

Since $\sim$ is an equivalence relation, $\mathcal{N}$ is partitioned into equivalence classes. If it can be shown that any arbitrary norm is equivalent to a specific fixed norm, then all norms belong to the equivalence class of the specific fixed norm, and therefore all norms are equivalent to each other.

Next, note that (1) holds trivially for $\mathbf{v} = 0$. Hence to show that two norms are equivalent it suffice to show that (1) holds for all $\mathbf{v} \in \mathrm{V}$ such that $\mathbf{v} \neq 0$. That is

$$m \leq \frac{\|\mathbf{v}\|'}{\|\mathbf{v}\|} \leq M \tag{5}$$

for all $\mathbf{v} \neq 0$. Since $\|\mathbf{v}\| > 0$, the scalar multiplication property of norms gives

$$\frac{\|\mathbf{v}\|'}{\|\mathbf{v}\|} = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\|' .$$

Hence $\|\cdot\|' \sim \|\cdot\|$ if and only if there exist real numbers m, M satisfying $0 < m \leq M < \infty$ such that

$$m \leq \|\mathbf{u}\|' \leq M \tag{6}$$

for all $\mathbf{u} \in \mathrm{V}$ such that $\|\mathbf{u}\| = 1$.

Now consider the problem at hand. The vector space $\mathbb{R}^n$ when equipped with the Euclidean norm, which is defined for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ by

$$\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}, \tag{7}$$

is called n-dimensional Euclidean space. In addition to being a vector space, Euclidean space is also a metric space with metric defined by $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, called the Euclidean metric, or Euclidean distance. The problem will be solved by exploiting a well-known property of Euclidean spaces to show that an arbitrary norm $\|\cdot\|$ is equivalent to $\|\cdot\|_2$, and thus all norms on $\mathbb{R}^n$ are equivalent.

To prove that all norms on $\mathbb{R}^n$ are equivalent it will suffice to show that there exist real numbers $0 < m \leq M < \infty$ such that $m \leq \|\mathbf{u}\| \leq M$ for all

$$\mathbf{u} \in S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\|_2 = 1\} .$$

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The set S^{n-1} is the unit sphere centered at the origin; it is bounded and closed. The Extreme Value Theorem (EVT) for Euclidean space guarantees that a continuous real-valued function on a closed and bounded set achieves both a minimum and a maximum value on the set. A norm by definition is a real-valued function. Thus if it is shown that an arbitrary norm is a continuous function with respect to the Euclidean metric, then by the EVT there exist $\mathbf{a}, \mathbf{b} \in S^{n-1}$ such that

$$\|\mathbf{a}\| \leq \|\mathbf{u}\| \leq \|\mathbf{b}\| . \tag{8}$$

Since $\mathbf{a} \in S^{n-1}$, $\mathbf{a} \neq 0$, and the norm of a vector is zero if and only if the vector is the zero vector, $\|\mathbf{a}\| > 0$. On the other hand, the norm of any vector is finite. Hence $\|\mathbf{b}\| < \infty$. Therefore if it is shown that an arbitrary norm is a continuous function, then (6) will hold with $m := \|\mathbf{a}\|$ and $M := \|\mathbf{b}\|$.

Thus to complete the proof it suffices to show that an arbitrary norm is a continuous function with respect to the Euclidean metric. To this end, the definition of continuity of a function between metric spaces is used. For arbitrary fixed $\mathbf{x} \in \mathbb{R}^n$ and any $\epsilon > 0$, a $\delta > 0$ is exhibited such that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| < \epsilon$$

whenever

$$\|\mathbf{x} - \mathbf{y}\|_2 < \delta.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ be the standard basis. An application of the reverse triangle inequality followed by an application of the triangle inequality gives

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\| = \left\| \sum_{\mathbf{i=1}}^{\mathbf{n}} (x_i - y_i)\mathbf{e}_i \right\| \leq \sum_{i=1}^{n} |x_i - y_i| \, \|\mathbf{e}_i\| . \tag{9}$$

Thus

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq n \max_{1 \leq i \leq n} |x_i - y_i| \max_{1 \leq i \leq n} \|\mathbf{e}_i\| . \tag{10}$$

Suppose that

$$\|\mathbf{x} - \mathbf{y}\|_2 < \frac{\epsilon}{n \max_{1 \leq i \leq n} \|\mathbf{e}_i\|} =: \delta. \tag{11}$$

For any $1 \leq i \leq n$,

$$|x_i - y_i| \leq \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} = \|\mathbf{x} - \mathbf{y}\|_2 . \tag{12}$$

The right hand side of this inequality does not depend on the index i. Hence if inequality (11) is satisfied, then

$$\max_{1 \leq i \leq n} |x_i - y_i| < \frac{\epsilon}{n \max_{1 \leq i \leq n} \|\mathbf{e}_i\|}, \tag{13}$$

and as a a consequence of the inequality (10)

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| < \epsilon, \tag{14}$$

which proves that $\|\cdot\|$ is continuous on $\mathbb{R}^n$ with respect to the Euclidean metric. ■