Featured Problem Series
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Week 4

Problem

This week our Featured Problem is from Penn State Math 403, an upper-level real analysis course. This challenging problem requires multi-step reasoning and a bit of creative topological insight. It's a great test of analytical persistence.

(a) Consider the following sets all equipped with the Euclidean metric:

- $\mathbb{R}^2$;
- $H := \{(x, y) \in \mathbb{R}^2 | y > 0\}$, the upper half-plane; and
- $L := \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 | x \geq 0\}$, the plane with the nonnegative x-axis removed.

Are any of these metric spaces homeomorphic?

(b) Are $\mathbb{Q}$ and $\mathbb{Z}$ both equipped with the absolute value metric homeomorphic?

Solution

The solution to this problem is rather long and involved, and the reader might lose sight of the big picture while navigating the details. To minimize this possibility a quick review is given to fix ideas, set notation, and make the strategy of the solution apparent.

Two metric spaces (X, d) and (Y, d') are homeomorphic, denoted $(X, d) \cong (Y, d')$ or simply $X \cong Y$, when no confusion about the metric is likely to arise, if there exits a continuous bijection $f|X \to Y$ with a continuous inverse. The map f is called a homeomorphism.

The direct way to prove that two metric spaces are homeomorphic is to exhibit a homeomorphism between the metric spaces. To prove that two metric spaces are not homeomorphic, it suffices to show that one metric space has a property preserved by homeomorphisms-such properties are called invariants-that the other space does not have. An indirect approach for showing two metric spaces are homeomorphic or not is to exploit the fact that $\cong$ is an equivalence relation. Therefore the collection of all metrics spaces is partitioned in to equivalence classes of homeomorphic metric spaces. Hence when considering three metric spaces, X, Y, and Z, if it is shown that $X \cong Y$, and $Y \cong Z$, it follows that $X \cong Z$. On the other hand, if it is shown that $X \cong Y$, and $Y \ncong Z$, then $X \ncong Z$.

Within the direct approach it will be necessary to prove that a map and its inverse are both continuous. There are several ways this can be done. The definition of continuity in a metric space will be used here: Suppose that (X, d) and (Y, d') are a pair of metric spaces. A function $f|X \to Y$ is continuous at $x \in X$ if given an $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d'\left(f(x), f(x')\right) < \epsilon \tag{1}$$

whenever

$$d\left(x, x'\right) < \delta. \tag{2}$$

A function is called continuous if it is continuous at all $x \in X$.

Next a bit of notation is introduced to distinguish between a point of $\mathbb{R}^2$ and the components of the points. To this end, write $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. In a similar way, a function with domain and codomain contained in $\mathbb{R}^2$ will be denoted by

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})) = (f_1(x_1, x_2), f_2(x_1, x_2)) \in \mathbb{R}^2.$$

Finally, recall the definition of Euclidean metric. For any $A \subseteq \mathbb{R}^2$, the Euclidean metrics is the function $\rho|A \times A \to [0, \infty)$ given by

$$\rho\left(\mathbf{x}, \mathbf{x}'\right) := \sqrt{\left(x_1 - x_1'\right)^2 + \left(x_2 - x_2'\right)^2} \tag{3}$$

for $\mathbf{x} = (x_1, x_2), \mathbf{x}' = (x_1', x_2') \in A$.

Proof of (a):

The direct approach is used to show that $\mathbb{R}^2 \cong H$. A combination of the direct and indirect approaches is used to show that and $H \cong L$. It will then follow from the indirect approach that $\mathbb{R}^2 \cong L$.

The construction of a homeomorphism $\mathbf{f}\big|\mathbb{R}^2 \to H$ starts with a few observations. The first is that that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $H = \mathbb{R} \times (0, \infty)$. Next, note that the maps $g|\mathbb{R} \to \mathbb{R}$ and $h|\mathbb{R} \to (0, \infty)$ given by $g(x) = x$ and $h(x) = e^x$, $x \in \mathbb{R}$ have inverses $g^{-1}\big|\mathbb{R} \to \mathbb{R}$ and $h^{-1}\big|(0, \infty) \to \mathbb{R}$ given by $g^{-1}(x) = x$ and

$h^{-1}(x') = \ln x'$ for $x \in \mathbb{R}$, and $x' \in (0, \infty)$. Since a function is a bijection if and only if it has an inverse, the function g and h are bijections. Further, as is typically shown in calculus I, if $\mathbb{R}$ and $(0, \infty)$ are both equipped with the absolute value metric, then g, h, g^{-1}, and h^{-1} are all continuous.

Define $\mathbf{f}\big|\mathbb{R}^2 \to H$ and $\mathbf{F}\big|H \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) =: (g(x_1), h(x_2)) = (x_1, e^{x_2}), \tag{4}$$

and

$$\mathbf{F}(\mathbf{x}) =: (g^{-1}(x_1), h^{-1}(x_2)) = (x_1, \ln x_2). \tag{5}$$

One has $\mathbf{F}\circ\mathbf{f}(x_1, x_2) = (x_1, x_2), (x_1, x_2) \in \mathbb{R}^2$ and $\mathbf{f}\circ\mathbf{F}(x_1, x_2) = (x_1, x_2), (x_1, x_2) \in H$. Thus $\mathbf{F} = \mathbf{f}^{-1}$. Consequently $\mathbf{f}$ is a bijection.

The continuity of g and h are leveraged to prove that $\mathbf{f}$ is a continuous function. Given an $\epsilon > 0$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ a $\delta > 0$ is exhibited such that

$$\rho\left(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}')\right) < \epsilon, \tag{6}$$

whenever

$$\rho\left(\mathbf{x}, \mathbf{x}'\right) < \delta. \tag{7}$$

By the continuity of g and h, given an $\frac{\epsilon}{\sqrt{2}} > 0$, $x_1, x_2 \in \mathbb{R}$ there exit $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|g(x_1) - g(x_1')| < \frac{\epsilon}{\sqrt{2}}, \qquad \text{and} \tag{8}$$

$$|h(x_2) - h(x_2')| < \frac{\epsilon}{\sqrt{2}} \tag{9}$$

whenever $|x_1 - x_1'| < \delta_1$ and $|x_2 - x_2'| < \delta_2$, respectively. Suppose that

$$\rho\left(\mathbf{x}, \mathbf{x}'\right) := \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2} < \delta := \min\{\delta_1, \delta_2\}. \tag{10}$$

It follows from (10), $(x_1 - x_1')^2, (x_2 - x_2')^2 \geq 0$, and that the fact that the square root is an increasing function that

$$|x_1 - x_1'| \leq \rho\left(\mathbf{x}, \mathbf{x}'\right) < \delta \leq \delta_1, \tag{11}$$

and

$$|x_2 - x_2'| \leq \rho\left(\mathbf{x}, \mathbf{x}'\right) < \delta \leq \delta_2. \tag{12}$$

Consequently the bounds (8) and (9) hold. Thus

$$\begin{aligned} \rho\left(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}')\right) =& \sqrt{\left(g(x_1) - g(x_1')\right)^2 + \left(h(x_2) - h(x_2')\right)^2} \\ <& \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon, \end{aligned} \tag{13}$$

whenever

$$\rho\left(\mathbf{x},\mathbf{x}'\right)<\delta. \tag{14}$$

This proves the continuity of f.The proof that f^{-1} is also continuous is nearly identical, and left to the reader. Thus $\mathbf{f}\big|\mathbb{R}^2\to H$ is a homeomorphism, i.e. $\mathbb{R}^2\cong H$.

Both the direct approach and indirect approach are marshaled to prove that $H\cong L$. First, the metrics spaces (H,ρ) and (L,ρ) are shown to be homeomorphic to metric spaces defined in terms of polar coordinate. Then these metric spaces are shown to be homeomorphic to each other.

First, consider the sets

$$M=:\{(r,\theta)|0<r<\infty,0<\theta<\pi\}=\mathbb{R}^+\times(0,\pi) \tag{15}$$

and

$$N=:\{(r,\theta)|0<r<\infty,0<\theta<2\pi\}=\mathbb{R}^+\times(0,2\pi) \tag{16}$$

For $A\subseteq N$ define a metric $\sigma|A\times A\to[0,\infty)$ by

$$\sigma(\mathbf{r},\mathbf{R})=\sqrt{r^2+R^2-2rR\cos\left(\theta-\Theta\right)}, \tag{17}$$

where $\mathbf{r}=(r,\theta),\mathbf{R}=(R,\Theta)\in A$. Since $M\subset N$, σ is defined on $M\times M$, as well as $N\times N$. It is shown that $(H,\rho)\cong(M,\sigma)$ and $(L,\rho)\cong(N,\sigma)$.

A homeomorphism is between (L,d) and (N,σ) is exhibited. Define $T_1|L\to(0,\infty)$, $T_2|L\to(0,2\pi)$, $S_1|N\to L$, and $S_2|N\to L$ by

$$T_1(\mathbf{x}):=\sqrt{x_1^2+x_2^2}=:r \tag{18}$$

$$T_2(\mathbf{x}):=\begin{cases}\arccos\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right) & \text{for } x_2\geq 0\\ \arccos\left(-\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right)+\pi & \text{for } x_2<0.\end{cases} \tag{19}$$

$$S_1(\mathbf{r}):=r\cos\theta, \tag{20}$$

and

$$S_2(\mathbf{r}):=r\sin\theta. \tag{21}$$

Finally, define $\mathbf{T}|L\to N$ and $\mathbf{S}|N\to L$ by

$$\mathbf{T}(\mathbf{x}):=(T_1(\mathbf{x}),T_2(\mathbf{x})),$$

and

$$\mathbf{S}(\mathbf{r}):=(S_1(\mathbf{r}),S_2(\mathbf{r})).$$

Observe that $\mathbf{S}\circ\mathbf{T}(\mathbf{x})=\mathbf{x}$ and $\mathbf{T}\circ\mathbf{S}(\mathbf{r})=\mathbf{r}$. Thus $\mathbf{T}$ is invertible with inverse $\mathbf{T}^{-1}=\mathbf{S}$. Consequently $\mathbf{T}$ is a bijection.

In preparation for proving the continuity of $\mathbf{T}$ and $\mathbf{T}^{-1}$ a relation between ρ and σ is exhibited. For $\mathbf{x}=(x_1,x_2),\mathbf{x}'=(x_1',x_2')\in L$, set $\mathbf{r}=(r,\theta)=\mathbf{T}(\mathbf{x}),\mathbf{R}=(R,\Theta)=\mathbf{T}(\mathbf{x}')$, one has

$$\begin{aligned}\sigma(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{x}')) =&\sqrt{r^2+R^2-2rR\cos(\theta-\Theta)}\\ =&\sqrt{r^2+R^2-2rR\left[\cos\theta\cos\Theta+\sin\theta\sin\Theta\right]}\\ =&\sqrt{r^2+R^2-2x_1x_1'-2x_2x_2'}\\ =&\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}\\ =&\rho(\mathbf{x},\mathbf{x}'),\end{aligned} \tag{22}$$

where the second equality follows from the difference formula for the cosine,the third equality is a consequence of (20) and (21), and (18) gives the fourth equality. On the other hand, for $\mathbf{r},\mathbf{R}\in N$, set $\mathbf{x}=\mathbf{T}^{-1}(\mathbf{r})$ and $\mathbf{x}'=\mathbf{T}^{-1}(\mathbf{R})$. It follows from (22) that

$$\rho(\mathbf{T}^{-1}(\mathbf{r}),\mathbf{T}^{-1}(\mathbf{R}))=\rho(\mathbf{x},\mathbf{x}')=\sigma(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{x}'))=\sigma(\mathbf{r},\mathbf{R}). \tag{23}$$

With (22) and (23) in hand, it is straightforward to prove that both $\mathbf{T}$ and $\mathbf{T}^{-1}$ are continuous.

Given $\mathbf{x}\in L$ and an $\epsilon>0$, if $\rho(\mathbf{x},\mathbf{x}')<\delta:=\epsilon$, it follows from (22) that $\sigma(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{x}'))<\epsilon$. Thus $\mathbf{T}|L\to N$ is continuous. The continuity of $\mathbf{T}^{-1}\big|N\to L$ follows from a nearly identical argument that uses (23) in place of (22).

It has been shown that $T|L\to N$ is a homeomorphism, thus $L\cong N$. Further, since $H\subset L$ and $M\subset N$, $H\cong M$.

As an aside, a bijection between metric spaces which satisfies (22) is called an isometry, and the spaces are called isometric. The above argument shows that two isometric metric spaces are homeomorphic. The converse is not true.

Next, it is proven that $M\cong N$. To this end, consider the mapping $\boldsymbol{\phi}|M\to N$ given by

$$\boldsymbol{\phi}(\mathbf{r}):=(r,2\theta), \tag{24}$$

where $\mathbf{r}=(r,\theta)\in M$. Clearly, $\boldsymbol{\phi}^{-1}(\mathbf{r})=\left(r,\frac{\theta}{2}\right)$ for $\mathbf{r}=(r,\theta)\in N$. Thus $\boldsymbol{\phi}$ is a bijection.

Now it is argued that $\boldsymbol{\phi}$ is continuous. To this end, note that the half-angle formula and (17) give

$$\sigma(\mathbf{r},\mathbf{R})=\sqrt{(r-R)^2+4rR\sin^2\left(\frac{\theta-\Theta}{2}\right)}, \tag{25}$$

Given $\epsilon>0$ and $\mathbf{r}=(r,\theta)\in M$, a $\delta>0$ is exhibited such

$$\sigma(\boldsymbol{\phi}(\mathbf{r}),\boldsymbol{\phi}(\mathbf{R}))<\epsilon, \tag{26}$$

whenever

$$\sigma(\mathbf{r},\mathbf{R}) < \delta. \tag{27}$$

Set $\delta = \frac{\epsilon}{\sqrt{5}}$, Suppose that $\sigma(\mathbf{r},\mathbf{R}) < \delta$, then (25) and an argument similar to that used to derive the bounds (11) and (12) give

$$|r - R| \leq \sigma(\mathbf{r},\mathbf{R}) < \frac{\epsilon}{\sqrt{5}} \tag{28}$$

and

$$2\sqrt{rR}\left|\sin\left(\frac{\theta-\Theta}{2}\right)\right| \leq \sigma(\mathbf{r},\mathbf{R}) < \frac{\epsilon}{\sqrt{5}}. \tag{29}$$

If $\sigma(\mathbf{r},\mathbf{R}) < \delta$, then

$$\begin{aligned}\sigma(\boldsymbol{\phi}(\mathbf{r}),\boldsymbol{\phi}(\mathbf{R})) =& \sqrt{(r-R)^2 + 4rR\sin^2\left(\frac{2\theta-2\Theta}{2}\right)} \\ =& \sqrt{(r-R)^2 + 4rR\sin^2(\theta-\Theta)} \\ =& \sqrt{(r-R)^2 + 16rR\sin^2\left(\frac{\theta-\Theta}{2}\right)\cos^2\left(\frac{\theta-\Theta}{2}\right)} \\ \leq& \sqrt{(r-R)^2 + 16rR\sin^2\left(\frac{\theta-\Theta}{2}\right)} \\ <& \sqrt{\frac{\epsilon^2}{5} + \frac{4\epsilon^2}{5}} = \epsilon,\end{aligned} \tag{30}$$

where the third equality follows from the double angle formula, and the first inequality uses the fact that the cosine in absolute value is bounded above by 1.

Consequently

$$\sigma(\boldsymbol{\phi}(\mathbf{r}),\boldsymbol{\phi}(\mathbf{R})) < \epsilon$$

whenever

$$\sigma(\mathbf{r},\mathbf{R}) < \delta$$

Thus $\boldsymbol{\phi}|M \to N$ is continuous. A similar argument gives the continuity of $\boldsymbol{\phi}^{-1}\big|N \to M$. The reader can fill in the details of that argument.

It has been shown that $\boldsymbol{\phi}|M \to N$ is a homeomorphism, thus $M \cong N$. It was shown earlier $H \cong M$ and $L \cong N$. Therefore, by the indirect method $H \cong L$

It has also shown earlier that $\mathbb{R}^2 \cong H$, which coupled with $H \cong L$, implies via the indirect method that $\mathbb{R}^2 \cong L$. ■

Proof of (b):

It is proven that $\mathbb{Z} \ncong \mathbb{Q}$ as follows. First, it is shown that isolated points are invariant under a homeomorphism. Thus homeomorphic metric spaces must have the same number of isolated points. Next, it is shown that all points of $\mathbb{Z}$ with respect to the absolute value metric are isolated. Finally, it argued that none of the points of $\mathbb{Q}$ with respect to the absolute value metric are isolated. This lead to the conclusion that $\mathbb{Z} \ncong \mathbb{Q}$.

Before the proof is presented, a short review is given to fix ideas and notation.

Consider a metric space (X, d). An open ball centered at $x \in X$ of radius $r \geq 0$ is the set

$$\mathrm{B}_r(x) = \{x' \in X | d(x, x') < r\}.$$

A set $G \subseteq X$ is open if and only if for all $x \in G$ there exists an $r > 0$ such that $\mathrm{B}_r(x) \subseteq G$. Open balls are open sets. A point of $x \in X$ is isolated if and only if there exists a $r > 0$ such that $\mathrm{B}_r(x) \backslash \{x\} = \emptyset$.

A characterization of continuity which is equivalent to the definition given earlier will be used in the proof of (b). Given two metric spaces (X, d) and (Y, d'), a map $f|X \to Y$ is continuous if and only if for all open $G \subseteq Y$, $f^{-1}(G)$ is open in X.

Now suppose that $(X, d) \cong (Y, d')$ with homeomorphism f. It is shown that if x is an isolated point of X, then $y = f(x)$ is an isolated point of Y. Since $x \in X$ is an isolated point, for some $r > 0$, $\mathrm{B}_r(x) \backslash \{x\} = \emptyset$. Thus

$$f\left(\mathrm{B}_r(x) \backslash \{x\}\right) = f(\emptyset) = \emptyset. \tag{31}$$

A bijection is an injection. The image of an intersection under an injection is the intersection of the images. Further, $\mathrm{B}_r(x) \backslash \{x\} = \mathrm{B}_r(x) \cap \{x\}^c$. Therefore

$$\emptyset = f\left(\mathrm{B}_r(x) \backslash \{x\}\right) = f\left(\mathrm{B}_r(x)\right) \backslash f\left(\{x\}\right) = f\left(\mathrm{B}_r(x)\right) \backslash \{y\}. \tag{32}$$

The inverse function f^{-1} is continuous and $\mathrm{B}_r(x)$ is an open set in X, hence $(f^{-1})^{-1}\left(\mathrm{B}_r(x)\right) = f\left(\mathrm{B}_r(x)\right)$ is an open set in Y. Clearly, $y = f(x) \in f\left(\mathrm{B}_r(x)\right)$. Therefore there exists an $R > 0$ such that

$$\mathrm{B}'_R(y) = \{y' \in Y | d'(y, y') < R\} \subseteq f\left(\mathrm{B}_r(x)\right).$$

It follows that

$$\mathrm{B}'_R(y) \backslash \{y\} \subseteq f\left(\mathrm{B}_r(x) \backslash \{x\}\right) = \emptyset. \tag{33}$$

Consequently y is an isolated point of Y.

Let $A \subseteq \mathbb{R}$ be a metric space with the absolute value metric. An open ball in A centered at $x \in A$ of radius r is given by

$$\mathrm{B}^A_r(x) = (x - r, x + r) \cap A.$$

Now consider the open ball in $\mathbb{Z}$ of radius $r = \frac{1}{2}$ centered at any $x \in \mathbb{Z}$,

$$\mathrm{B}^{\mathbb{Z}}_{\frac{1}{2}}(x) = \left(x - \frac{1}{2}, x + \frac{1}{2}\right) \cap \mathbb{Z} = \{x\}. \tag{34}$$

Consequently $\mathrm{B}^{\mathbb{Z}}_{\frac{1}{2}}(x)\backslash\{x\} = \emptyset$ Thus all points of $\mathbb{Z}$ are isolated.

Next, consider $x \in \mathbb{Q}$. One has

$$\mathrm{B}^{\mathbb{Q}}_{r}(x) = (x - r, x + r) \cap \mathbb{Q}. \tag{35}$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ every open interval in $\mathbb{R}$ contains a countably infinite number of points in $\mathbb{Q}$. Thus for all $r > 0$, $\mathrm{B}^{\mathbb{Q}}_{r}(x)$ contains at least one point other than x. Therefore for all $r > 0$, $\mathrm{B}^{\mathbb{Q}}_{r}(x)\backslash\{x\} \neq \emptyset$. It follows that $\mathbb{Q}$ has no isolated points with respect to the absolute value metric. ■

A short endnote is in order. Anyone interested in exploring part (b) in more depth may want to look into Sierpinski's Theorem, which is not typically taught in an upper-level real analysis course. It characterizes all metric spaces which are homeomorphic to $\mathbb{Q}$ with the absolute value metric.

Theorem 1 (Sierpinki)**.** *Any countably infinite metric space without isolated points is homeomorphic to $\mathbb{Q}$ equipped with the absolute value metric.*