Featured Problem Series
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Week 3

Problem

The problem this week comes from Penn State Math 414, an upper-level non measure-theoretic probability course.

A card game consists of a dealer, a player, and a deck of n cards numbered 1 through n. The dealer shuffles the deck so that all $n!$ permutations of the cards are equally likely. The game starts with the player guessing the top card in the deck. The dealer removes the top card from the deck, examines it, and tells the player only if the guess was correct or not. The game continues with the player guessing the new top card, the dealer removing it, examining it, and informing the player if the guess was correct or not. The game ends after all n cards have been removed from the deck.

When the player's guess is correct the player knows that that card has been removed from the deck, but when the guess is incorrect the player does not know the card that has been removed. With this in mind, the player adopts the strategy of guessing the same card as at the previous step, until either the guess is correct or the game ends. When a guess is correct, the player changes the card guessed at the next step to one of the cards not already guessed at previous steps, since the player knows those cards are no longer in the deck. Let N denote the number of correct guesses under this strategy. Find $\mathbf{E}[N]$.

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The Solution

The following identity will be used in the solution.

Proposition 1. *If X is a nonnegative random variable, then*

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}[X > x] \, \mathrm{d}x, \tag{1}$$

Further, if X takes on values in $\mathbb{N}$, then

$$\mathbf{E}[X] = \sum_{x=1}^\infty \mathbf{P}[X \geq x]. \tag{2}$$

The random variable N takes on values in $\mathbb{N}$ with a maximum possible value of n. Thus $\mathbf{P}[N \geq r] = 0$, for $r > n$, and

$$\mathbf{E}[N] = \sum_{r=1}^n \mathbf{P}[N \geq r] \tag{3}$$

The problem reduces to finding $\mathbf{P}[N \geq r]$ the probability of at least r correct guesses for $r = 1, 2, \ldots, n$.

To get a handle on how to compute these probabilities the guessing strategy is examined more closely. The guessing strategy, at least superficially, appears to be dynamic. After a correct guess the player makes a decision about the card to guess next. However, the player has no more knowledge about which card to switch to after the correct guess,than before the correct guess, since the only information the player has to base the next guess on is the cards not yet guessed, and the player possesses this information prior to the unveiling of the correct guess, as well as afterward. Thus the player could inform the dealer at the outset that the first guess is g_1, and his new guess after the $i-1$ correct guess is $g_i, i = 2, \ldots, n$. That is the player gives a the dealer some permutation of the first n natural numbers $g_1, g_2, \ldots, g_n$ with the instruction that that after each correct guess the new guess is the next number in the permutation.

The number of correct guesses will be at least $1 \leq r \leq n$ if the sequence $g_1, g_2, \ldots g_r$ appears in that order, not necessarily adjacent to each other, in the deck of cards. Of the $n!$ possible orderings of the deck, the number of orderings which contain the sequence in the given order is

$$\binom{n}{r} \cdot 1 \cdot (n-r)! \tag{4}$$

The first factor is the number of ways to select$r$ location in the deck to receive the cards in the sequence, the second factor represents the single way to arrange the specific cards $g_1, g_2, \ldots, g_r$ in their predetermined order within those r selected locations, and the final factor is the number of ways to permute the remaining cards in the remaining locations. Consequently

$$\mathbf{P}[N \geq r] = \frac{\binom{n}{r}(n-r)!}{n!} = \frac{1}{r!}. \tag{5}$$

Therefore

$$\mathbf{E}[N] = \sum_{r=1}^{n} \frac{1}{r!}. \tag{6}$$

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