Featured Problem Series
Fall 2025

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Week 2

Problem

This week there are two problems both from discrete math, such as one one might find in Penn State Math 311W. What links these problems, and why we chose them, is that they both have the flavor of a Paul Erdős problem, but they are by no means as difficult as an Erdős problem.

For those of you who are unaware, Paul Erdős (1913-1996) was an itinerant Hungarian mathematician known for problems which were simple to state and difficult to solve, and a legion of collaborators. His prolific collaborations have lead to the introduction of Erdős numbers which measures ones "collaborative distance" from him. These numbers are defined recursively. Erdős himself has an Erdős number of zero. Anyone who has co-authored a paper with him has an Erdős number of one. The Erdős number two is assigned to anyone who has not written a paper with Erdős, but has written a paper with a person with Erdős number one. Or in general, a person has Erdős number $n+1$ if the lowest Erdős number among their collaborators is n. There are 511 people with Erdős number one. We are one of the 12,500 people who have an Erdős number two.

Without further adieu here are this week's problems.

(a) Prove that any subset of $\{1, 2, \ldots, 200\}$ size 101 contains at least two consecutive integers.

(b) Let $S = \{1, 2, \ldots, 2n+1\}, n \in \mathbb{N}$. Prove that any subset of S of size $n+2$ contains two elements that sum to $2n+2$.

Solution

Proof of (a). The proof goes by contradiction. Let $A \subset S$ with $|A| = 101$. The elements of A are arranged in increasing order $1 \leq n_0 < n_1 < \cdots < n_{100} \leq 200$. Set $g_i = n_i - n_{i-1}, i = 1, 2, \ldots 100$. Note that $g_i \geq 1$ for $i = 1, 2, \ldots, 100$ and the integers n_i and n_{i-1} are consecutive if $g_i = 1$. One has

$$\begin{aligned} \sum_{i=1}^{100} g_i &= \sum_{i=1}^{100} (n_i - n_{i-1}) = \sum_{i=1}^{100} n_i - \sum_{i=1}^{100} n_{i-1} \\ &= \sum_{i=1}^{100} n_i - \sum_{i=0}^{99} n_i = n_{100} - n_0. \end{aligned} \tag{1}$$

Thus

$$n_{100} = n_0 + \sum_{i=1}^{100} g_i. \tag{2}$$

Suppose that there are no consecutive integers in A, then $g_i \geq 2$ for all i. It follows that

$$n_{100} = n_0 + \sum_{i=1}^{100} g_i \geq n_0 + 2 \cdot 100 \geq 1 + 200 = 201.$$

This contradicts $n_{100} \leq 200$. Therefore there must be at least one pair of consecutive integers in A. ■

Proof of (b). It shown that the largest subset which contains no pairs of elements summing to $2n+2$ has $n+1$ elements. There are n pairs of elements in S which sum to $2n+2$, these are

$$(k, 2n+2-k), \quad k = 1, 2, \ldots, n. \tag{3}$$

Observe that the element $n+1$ does not appear in any of these pairs, since it would have to pair with itself to sum to $2n+2$. Consequently a subset which does not contain any pairs summing to $2n+2$ can have at most one element from each of the n pairs and the element $n+1$. Hence the maximum size of such a subset is $n+1$. It follows that if a subset has $n+2$ elements there must be at least one pair of elements in the subset summing to $2n+2$. ■

Remark: The solution to part (b) implicitly uses the Pigeonhole Principle: If you have n Pigeonholes and $n+1$ pigeons, at least one hole must have two pigeons. This principle can also be used to prove part (a).We will leave it to the reader to make the use of this principle explicit in the proof of (b), and to devise an alternative proof of (a) that uses the principle.