Problem of the Week
Fall 2025

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Week 1

Problem

The Fall Featured Problem Series kicks off today with a sophomore-level real analysis problem like one you may encounter in Penn State Math 312. However, we think the solution should be within the reach of a strong calculus II student.

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfying $a_n \neq 0$ for all n, and that $\lim_{n\to\infty} a_n = A$ with $0 < |A| < \infty$. Prove that the two series

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left| \frac{1}{a_{n+1}} - \frac{1}{a_n} \right| \tag{1}$$

both either converge or diverge.

Solution

The problem asks to prove that two series both either converge or diverge. which makes the Limit Comparison Test (LCT) the natural choice for the solution, though, for a technical reason discussed below in a remark, it is not applicable to this problem. However, the Direct Comparison Test (DCT) can be used. Recall the statement of the DCT: Consider two series $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$, such that

$$0 \le b_i \le c_i, \quad i = 1, 2, \dots, \tag{2}$$

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If $\sum_{i=1}^{\infty} c_i$ converges, then $\sum_{i=1}^{\infty} b_i$ converges. On the other hand, if $\sum_{i=1}^{\infty} b_i$ diverges, then $\sum_{i=1}^{\infty} c_i$ diverges.

Before the DCT is applied to this problem, observe that the condition $a_n \neq 0$ for all n gives $|a_n|^{-1} < \infty$ for all n. Thus the series on the right in (1) is well-defined.

The first step in the application of the DCT is to write the summand on the right in (1) in terms of of the summand on the left by obtaining a common denominator,

$$\left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| = \frac{|a_{n+1} - a_n|}{|a_{n+1}a_n|}, \quad n = 1, 2, \ldots. \tag{3}$$

Note that the assumption that the sequence $\{a_n\}_{n=1}^{\infty}$ converges, implies that the sequence is bounded, that is there exists a finite M such that $|a_n| \leq M$ for $n = 1, 2, \ldots$. And, the assumptions $A > 0$ and $a_n \neq 0$ for all n imply the existence of a positive m such that $m \leq |a_n|, n = 1, 2, \ldots$. These bounds along with (3) yield

$$\frac{1}{M^2}|a_{n+1} - a_n| \leq \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right| \leq \frac{1}{m^2}|a_{n+1} - a_n|, \tag{4}$$

$n = 1, 2, \ldots$. Multiplying all terms of a series by a nonzero constant does not change a convergent series to a divergent series, or a divergent series to a convergent series. Consequently it follows from the upper bound in (4) and the DCT, that if

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n|$$

converges, then

$$\sum_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|$$

converges. The lower bound in (4), and the DCT imply that if

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n|$$

diverges, then

$$\sum_{n=1}^{\infty} \left|\frac{1}{a_{n+1}} - \frac{1}{a_n}\right|$$

diverges. If follow that both series in (1) either converge or diverge. ■

Remark 1. Recall the LCT: Given $\sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty} c_i$, with $b_i \geq 0, c_i > 0$ for all i, if

$$\lim_{i \to \infty} \frac{b_i}{c_i} = L, \tag{5}$$

and $0 < L < \infty$, then both series either converge or diverge. This test appears to be the natural test to use in this problem, since the result will follow directly

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from verification of a single condition given in (5). But this overlooks the condition $c_i > 0$ for all i. The assumptions on $\{a_n\}_{n=1}^{\infty}$ do not preclude $a_{n+1} = a_n$ infinitely often. If this were the case, then both summands in (1) would be zero infinitely often, and neither summand could play the role of c_i in the application of the LCT.