Fall 2025
Featured Problem Series

R.J. Serinko, Ph.D.
Facebook ★ Reddit ★ YouTube ★ Mathstodon ★ Bluesky
https://imathtutor.org
rege@imathtutor.org
(814) 317-6284

Week 10

Problem

Our final problem in the Fall 2025 Featured Problem Series comes from Penn State's Math/Stat 416, an upper-level undergraduate stochastic modeling course. The challenge in solving this problem is to identify the appropriate model and corresponding method of solution.

Given three urns that initially contain two ball each, consider the following discrete time process. At each time step, an urn is selected with equal probability, and a ball is removed from that urn and placed with equal probabilities into one of the other urns. The process stops whenever an urn is empty. Find the expected number of steps before the process stops.

Solution

First a model of the process is constructed. This begins with the identification of an appropriate state space. The system is fully specified by the number of balls in each urn, thus the natural choice for a state space is

$$S = \left\{ (n_1, n_2, n_3) \middle| 0 \leq n_i \leq 5, \text{ and } \sum_{i=1}^{3} n_i = 6 \right\}, \tag{1}$$

where n_i denotes the number of balls in urn $i = 1, 2, 3$. The only thing in this choice of state space that may not be obvious is the upper bound on the

http://imathtutor.org
rege@imathtutor.org
(814) 317-6284

components of the vectors. For the system to transition to a state with six balls in one urn and none in the others, it must be in a state with one empty urn. This transition is not possible, since the process stops once an urn is empty. Thus the bound $n_i \leq 5, i = 1, 2, 3$.

The state of the system at time j is denoted by $\mathbf{N}_j \in S$ $j = 0, 1, \ldots$.

Next the dynamics of the transition from state to state at each time step is modeled. Let $\mathbf{X}_i = (X_{i,1}, X_{i,2}, X_{i,3})$ denote sequence of *i.i.d.* random vectors that take values in

$$\Delta = \left\{ (x_1, x_2, x_3) \middle| -1 \leq x_i \leq 1, i = 1, 2, 3, \prod_{i=1}^{3} (1 - x_i) = 0 = \sum_{i=1}^{3} x_i \right\} \tag{2}$$

with equal probability. The sequence of random vectors $\mathbf{X}_1, \mathbf{X}_2 \ldots$ models the process of moving a ball from one urn to another at each time step. For example, $\mathbf{X}_1 = (-1, 0, 1)$ corresponds to moving a ball from urn 1 to urn 3 at time step 1. At each step, a ball is always moved from one urn to another urn, hence the point $(0, 0, 0)$ is excluded from Δ by the condition $\prod_{i=1}^{3}(1 - x_i) = 0$.

The time evolution of the system is given by

$$\mathbf{N}_j = \mathbf{N}_{j-1} + \mathbf{X}_j \left[1 - I_{\{0\}} \left(\prod_{i=1}^{3} N_{(j-1),i} \right) \right]. \tag{3}$$

where $\mathbf{N}_0 = (2, 2, 2)$, and $I_{\{0\}}$ is the indicator function for the set $\{0\}$. Since the right hand side of (3) is a function of only the immediate past state of the system, and a random variable from an *i.i.d* sequence, it follows from a well-known result, which is presented at the end of this write-up for completeness, that the sequence $\mathbf{N}_0, \mathbf{N}_1 \ldots$ is a Markov chain with stationary transition probabilities. Further, the second factor in the second term on the right is equal to 1 if $\mathbf{N}_{j-1}$ has no zero components, and zero otherwise. Consequently any state with a zero component is absorbing, and once entered into the process stops evolving. Thus the problem reduces to finding the time before absorption for a stationary, absorbing Markov chain for which there is a well-developed method, which is now reviewed.

Consider a Markov chain $Y_1, Y_2 \ldots$ with stationary transition probabilities, and state space $S' = \{1, 2 \ldots, N\}$. The time-evolution of the state space distribution of the chain is fully specified by the stochastic matrix $\mathbf{P}$ which has $r, s \in S$ element

$$P_{rs} = \mathbf{P}\left[Y_1 = s | X_0 = r\right]. \tag{4}$$

Further, suppose that the states $1 \leq k \leq t$, $1 < t < N$ are transient and the remaining $a = N - t$ state are absorbing. Then $\mathbf{P}$ may be written as a partitioned matrix as follow

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I}_a \end{pmatrix}, \tag{5}$$

where $\mathbf{Q}$ is an $t \times t$ matrix with the transition probabilities between transient states, $\mathbf{R}$ is a $t \times a$ matrix with the transition probabilities between the transient

http://imathtutor.org
rege@imathtutor.org
(814) 317-6284

states and absorbing state, $\mathbf{0}$ is an $a \times t$ zero matrix, $\mathbf{I}_a$ is an $a \times a$ identity matrix. Many of the properties of interest for a stationary absorbing Markov chain, including the expected time spent in transient states before absorption when starting in transient state k, can be computed from the Fundamental matrix

$$\mathbf{F} := \mathbf{I}_t + \mathbf{Q} + \mathbf{Q}^2 + \cdots = (\mathbf{I}_t - \mathbf{Q})^{-1}. \tag{6}$$

Set

$$\mathbf{t} := \mathbf{F}\mathbf{1}, \tag{7}$$

where $\mathbf{1}$ is a $t \times 1$ matrix of ones. The k^{th} row of $\mathbf{t}$ is gives the expected number of steps before absorption when the chain starts in transient state k.

This method requires inverting an $t \times t$ matrix where t is the number of transient states of the process. The transient states of the chain $\mathbf{N}_0, \mathbf{N}_1, \dots$ are the states satisfying $n_1 + n_2 + n_3 = 6, 0 < n_i \leq 5$. The number of such states in S is given by

$$t = \binom{6-1}{3-1} = \binom{5}{2} = 10.$$

A matrix of this dimension will be tedious to invert by hand. One option would be to use software such as Matlab or R to invert the matrix. However, that will not be necessary, since the Markov process $\mathbf{N}_0, \mathbf{N}_1, \dots$ can be mapped on to a Markov process with a smaller, manageable state space.

Let $\Sigma = \{0, 1, 2, 3\}$ and define $f | S \to \Sigma$ by

$$f(\mathbf{n}) = \left[\sum_{j=1}^{3} I_{\{1\}}(n_j)\right] \left[1 - I_{\{0\}}\left(\prod_{i=1}^{3} n_i\right)\right] + 3I_{\{0\}}\left(\prod_{i=1}^{3} n_i\right). \tag{8}$$

Note that f maps a transient state on to its number of unit components, and all absorbing states onto 3.

Set $Y_j = f(\mathbf{N}_j), j = 0, 1, \dots$. Such a many-to-one map is called a course **graining map**, and the process $Y_0, Y_1, \dots$ is called **a course grained process**.

While $\mathbf{N}_0, \mathbf{N}_1, \dots$ is a Markov chain, it is not necessarily true that the course grained processes is a Markov chains. A sufficient condition for the course grained process to be a Markovian is for all $r, s \in \Sigma$.

$$\mathbf{P}\left[\mathbf{N}_{j+1} \in f^{-1}(\{r\}) \middle| \mathbf{N}_j = \mathbf{n}\right] = \mathbf{P}\left[\mathbf{N}_{j+1} \in f^{-1}(\{r\}) \middle| \mathbf{N}_j = \mathbf{n}'\right] \tag{9}$$

if $\mathbf{n}, \mathbf{n}' \in f^{-1}(\{s\})$. That is given that the original chain starts in a state that maps to s, the probability of transitioning to a state in $f^{-1}(\{r\})$ does not depend on the particular state in $f^{-1}(\{s\})$. Since the the chain $\mathbf{N}_0, \mathbf{N}_1, \dots$ have transition probabilities that do not depend on the time step, it will suffice to check the condition for $j = 0$.

Verification of (9) is straightforward. It starts with the following observa-

http://imathtutor.org
rege@imathtutor.org
(814) 317-6284

tions,

$$
\begin{aligned}
f^{-1}(\{0\}) =& \{(2,2,2)\}\\
f^{-1}(\{1\}) =& \{(1,3,2),(1,2,3),(3,1,2),(2,1,3),(2,3,1),(3,2,1)\}\\
f^{-1}(\{2\}) =& \{(1,1,4),(1,4,1),(4,1,1)\}\\
f^{-1}(\{3\}) =& \{(0,3,3),(0,4,2),(0,2,4),(3,0,3),(4,0,2),(2,0,4),(3,3,0),(4,2,0),\\
& (2,4,0),(0,1,5),(0,5,1),(1,0,5),(5,0,1),(1,5,0),(5,1,0)\}.
\end{aligned}
$$

Since $f^{-1}(\{0\})$ is a singleton, (9) holds trivially for an initial state in this set. Verification of (9) for the other cases uses the fact that $|\Delta| = 6$, and that all possible transitions are equally likely.

The condition (9) is verified for the case of the initial state in $f^{-1}(\{1\})$. The verification of the other cases is left to the reader. Each state in $f^{-1}(\{1\})$ can transition to a state in $f^{-1}(\{0\})$ in exactly one way; remove a ball from the urn with 3 balls and place it in the urn with 1 ball. Thus the probability for any state in $f^{-1}(\{1\})$ to transition to a state in $f^{-1}(\{0\})$ is $\frac{1}{6}$. It is also the case that each state in $f^{-1}(\{1\})$ can transition to a state in $f^{-1}(\{2\})$ in exactly one way: remove a ball from the urn with 2 balls and place it in the urn with 3 balls. Hence the probability for any state in $f^{-1}(\{1\})$ to transition to a state in $f^{-1}(\{2\})$ is also $\frac{1}{6}$. On the other hand, each state in $f^{-1}(\{1\})$ can transition to a state in $f^{-1}(\{3\})$ in two different ways: remove a ball from the urn with 1 ball and place it in either of the other two urns. Therefore the probability of transition to a state in $f^{-1}(\{3\})$ from any state in $f^{-1}(\{1\})$ is $\frac{2}{6}$. Finally, each state in $f^{-1}(\{1\})$ can transition to a state in $f^{-1}(\{1\})$ two different ways: remove a ball from the urn with 2 balls and place it in the urn with 1 ball, or remove a ball from the urn with 3 balls and place it in the urn with 2 balls. Consequently the probability of staying in $f^{-1}(\{1\})$ starting from any state in $f^{-1}(\{1\})$ is $\frac{2}{6}$.

The Markov chain $Y_j, j = 0, 1\ldots$ only has three transient states, 0,1,2. Hence one will only need to invert a 3×3 matrix to compute the Fundamental Matrix. which can be done by hand without out much effort.

Verification of (9) for all possible initial states yields the transition matrix $\mathbf{T}$ for the Markov chain $Y_0, Y_1, \ldots$

$$
\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{10}
$$

It follows that

$$
\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 \end{bmatrix}. \tag{11}
$$

http://imathtutor.org
rege@imathtutor.org
(814) 317-6284

Thus

$$\mathbf{I}_3 - \mathbf{Q} = \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}. \tag{12}$$

Gaussian reduction or Cramer's Rule can be used to find the inverse which is given by

$$\mathbf{F} = (\mathbf{I}_3 - \mathbf{Q})^{-1} = \begin{bmatrix} \frac{11}{8} & \frac{9}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{9}{4} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{4} & \frac{9}{8} \end{bmatrix}. \tag{13}$$

Finally, one has

$$\mathbf{F1} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}. \tag{14}$$

Thus the expected number of time steps before an urn is emptied given that the all three urns initially have two balls each is 4. ■

Theorem 1. *Let $f|\mathrm{X} \times \mathrm{Y} \to \mathrm{X}$. Suppose that X_0 is a random variable taking values in* X, *and $Y_1, Y_2, \ldots$ is an i.i.d. sequence of random variable taking values in* Y *that is independent of X_0. Then the sequence $X_0, X_1, \ldots$ defined by*

$$X_n = f(X_{n-1}, Y_n), \quad n \geq 1 \tag{15}$$

is a Markov process with state space X, *and stationary transition probabilities. In particular, this result holds if X_0 is degenerate, i.e. is a constant.*